6.1 Q7

If $f'(c)=0$ i.e. $\lim\limits_{x\to c} \dfrac{f(x)-f(c)}{x-c} = 0$

Then for any $\varepsilon>0$, there exists $\delta>0$ such that for any $x\in(c-\delta, c+\delta)$,

$$\left|\frac{g(x)-g(c)}{x-c}\right| = \left|\frac{|f(x)|-|f(c)|}{x-c}\right| = \frac{|f(x)|}{|x-c|} = \left|\frac{f(x)-f(c)}{x-c}\right| < \varepsilon$$

On the other hand, if $f'(c)=L\neq 0$ i.e. $\lim\limits_{x\to c}\dfrac{f(x)-f(c)}{x-c} = L$

Since $\left|\left|\frac{f(x)}{x-c}\right| - |L|\right| \leq \left|\frac{f(x)}{x-c} - L\right|$, $\lim\limits_{x\to c}\left|\frac{f(x)}{x-c}\right| = |L|$

We wish to apply Sequential Criterion to show

$$\lim_{x\to c}\frac{g(x)-g(c)}{x-c} = \lim_{x\to c}\frac{|f(x)|}{x-c} \quad \text{does not exist}$$

Take $x_n = c+\frac{1}{n}$, since $x_n\to c$ and $x_n>c$, then

$$\frac{|f(x_n)|}{x_n-c} = \left|\frac{f(x_n)}{x_n-c}\right| = |L| \quad \text{as } n\to\infty$$

Take $y_n = c-\frac{1}{n}$, since $y_n\to c$ and $y_n<c$, then

$$\frac{|f(y_n)|}{y_n-c} = -\left|\frac{f(y_n)}{y_n-c}\right| = -|L|$$

If $M := \lim\limits_{x\to c}\dfrac{|f(x)|}{x-c}$ exists, then $|L| = M = -|L|$

But $L\neq 0$ implies $|L|\neq -|L|$, we arrived at a contradiction!

6.1 Q12

Rmk: Since x^r is undefined for $x<0$, we consider the domain of f to be $[0,+\infty)$

The difference quotient at 0 is given by

$$\frac{f(x)-f(0)}{x-0} = \frac{x^r \sin(\frac{1}{x})}{x} = x^{r-1}\sin(\frac{1}{x})$$

Hence, $-x^{r-1} \leq \left| \frac{f(x)-f(0)}{x-0} \right| \leq x^{r-1}$ for $x \neq 0$

If $r>1$, we can apply squeeze theorem to show that

$$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0$$

If $0<r\leq 1$, consider two sequences $\{x_n\}$ and $\{y_n\}$ defined by

$$x_n = \frac{1}{2\pi n + \frac{\pi}{2}} \qquad y_n = \frac{1}{2\pi n}$$

When $r=1$, $\frac{f(x_n)-f(0)}{x_n-0} = 1$, $\frac{f(y_n)-f(0)}{y_n-0} = 0$

So the limit does not exist

When $0<r<1$, $\frac{f(x_n)-f(0)}{x_n-0} = (2\pi n+\frac{\pi}{2})^{1-r}$ is unbounded

The limit does not exist

In conclusion, we show that $f'(0)$ exists if and only if $r>1$

6.1 Q13

Note that if $f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ exists

As $\lim_{n\to\infty} \frac{1}{n} = 0$ and by Sequential Criterion, we have

$$f'(c) = \lim_{n\to\infty} \frac{f(c+\frac{1}{n}) - f(c)}{\frac{1}{n}} = \lim_{n\to\infty} \left(n\left(f(c+\tfrac{1}{n}) - f(c)\right)\right)$$

Here are two examples showing the existence of sequential limit does not imply the existence of $f'(c)$

Example 1 : Take $f(x) = |x|$ on $\mathbb{R}$ and $c=0$

It's clear that f is not differentiable at c

However, we still have $\lim_{n\to\infty} n(f(c+\frac{1}{n}) - f(c)) = \lim_{n\to\infty} n(|\frac{1}{n}| - 0) = 1$

Example 2 : Let $A = \{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$

Take f to be the characteristic function of A, χ_A

i.e. $\chi_A = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$

Take $c=0$, and it's clear that f is not differentiable at c

However, we still have $\lim_{n\to\infty} n(f(c+\frac{1}{n}) - f(c)) = \lim_{n\to\infty} n(1-1) = 0$

6.2 Q5

Let $f: [1, \infty) \to \mathbb{R}$ be defined by $f(x) = x^{\frac{1}{n}} - (x-1)^{\frac{1}{n}}$

We proceed to show that $f'(x) < 0$ for all $x \in (1, +\infty)$

Note that $f'(x) = \frac{1}{n} x^{\frac{1-n}{n}} - \frac{1}{n}(x-1)^{\frac{1-n}{n}}$

Since $n \geq 2$, $\frac{1-n}{n} < 0$

For all $x \in (1, \infty)$, $x > x-1 \geq 0$

Hence, $f'(x) = x^{\frac{1-n}{n}} - (x-1)^{\frac{1-n}{n}} < 0$

Finally, we need to check $f(\frac{a}{b}) < f(1)$

Since f is continuous on $[0, \frac{a}{b}]$ and differentiable on $(0, \frac{a}{b})$

By MVT, there exists $\xi \in (0, \frac{a}{b})$ such that

$$f(\tfrac{a}{b}) - f(1) = f'(\xi)(\tfrac{a}{b} - 1)$$

Since $f'(\xi) < 0$ and $\frac{a}{b} - 1 > 0$, we have $f(\frac{a}{b}) - f(1) < 0$

Now the result follows by noting $f(\frac{a}{b}) < f(1) \Leftarrow\Rightarrow a^{\frac{1}{n}} - b^{\frac{1}{n}} < (a-b)^{\frac{1}{n}}$

6.2 Q14

Suppose not, then there exists x, y such that $f'(x) < 0$, $f'(y) > 0$

Without loss of generality, suppose $x < y$

Since f is continuous on $[x, y]$, which is a compact set

By Extreme Value Theorem, f attains minimum at $x_0 \in [x, y]$

We proceed to show that x_0 in fact lies in (x, y) and

hence in the interior of I

Suppose it were true that $x_0 = x$

Since $f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} < 0$, then there exists $r > 0$

such that $\frac{f(t) - f(x)}{t - x} < 0$ for all $t \in I \cap (x-r, x+r) \setminus \{x\}$

In particular, when $t > x$ and $|x - t| < r$, we have $f(t) < f(x) = f(x_0)$

Contradict to the fact that $x_0 = x$ is the minimality

Similarly, f is locally strictly increasing at y

So it is impossible that $x_0 = y$

We must have $x_0 \in (x, y)$, so it lies in an open interval on which f is differentiable

By the Interior Extremum Theorem, an interior local extreme point has vanishing derivative, so $f'(x_0) = 0$, which is a contradiction to the assumption

The case $x > y$ could be done similarly by considering a local maximum.

6.2 Q17

Define $h(x) := f(x) - g(x)$, then h is also differentiable on $\mathbb{R}$

Since $f(0) = g(0)$ and $f'(x) \le g'(x)$ for all $x \ge 0$

Then $h(0) = f(0) - g(0) = 0$ and $h'(x) = f'(x) - g'(x) \le 0$ for all $x \ge 0$

It follows that $h(x) \le 0$ for all $x \ge 0$, which is equivalent to $f(x) \le g(x)$ for all $x \ge 0$